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Ramsey numbers for tournaments

Yannis Manoussakis[a,*], Zsolt Tuza[b,1]

[a] *LRI, Bât. 490, Université de Paris-Sud (Paris - XI), CNRS U.R.a. 410, F-91405 Orsay Cedex, France*
[b] *Computer and Automation Institute, Hungarian Academy of Sciences, H-1111 Budapest, Kende u. 13-17, Hungary*

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Abstract

The Ramsey number $r(D_1,\ldots,D_k)$ of *acyclic* directed graphs $D_1,\ldots,D_k$ is defined as the largest integer r for which there exists a tournament $\mathsf{T}=(V,A)$ on r vertices with a k-coloring $\varphi: A \to \{1,\ldots,k\}$ of the arc set A such that no D_i occurs in color i for any $i \in \{1,\ldots,k\}$. We discuss recursive techniques to compute $r(D_1,\ldots,D_k)$ in the case where there are paths and/or stars among the D_i. In particular, solving a problem of Bialostocki and Dierker [Congr. Numer. 47 (1985) 119–123], we prove that $r(D_1,D_2)=r(D_1)\cdot r(D_2)$ holds if D_1 is transitive and $D_2=\mathsf{S}_n$ is an out-going star on n vertices. Our main result is an asymptotic formula for $r(D_1,\ldots,D_k,\mathsf{S}_n)$ where the digraphs $D_1,\ldots,D_k$ are fixed arbitrarily and $n\to\infty$.

Keywords: Tournament; Ramsey number

1. Introduction

Let $D_1,\ldots,D_k$ be *acyclic* directed graphs (some or all may be identical). We define the k-color *Ramsey number* $r(D_1,\ldots,D_k)$ as the largest integer r for which there exists a tournament $\mathsf{T}=(V,A)$ with an r-element vertex set V and with a k-coloring $\varphi: A\to\{1,\ldots,k\}$ on its arc set A such that no D_i is a subdigraph of T in color i $(1\leqslant i\leqslant k)$. In the 'diagonal case', i.e., where all the D_i are isomorphic to the same digraph D, we shall write $r_k(D)$ for $r(D_1,\ldots,D_k)$; in particular, $r_1(D)$ and $r(D)$ mean the same. Moreover, for some special types of tournaments, we shall use the following notation:

* Corresponding author.
E-mail addresses: yannis@lri.fr (Y. Manoussakis), tuza@sztaki.hu (Z. Tuza).
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- TT_n – the transitive tournament of order n, with vertex set $\{v_1,\dots,v_n\}$ and arc set $\{v_iv_j \mid 1 \leqslant i < j \leqslant n\}$.
- P_n – the directed path of length $n-1$, with vertex set $\{v_1,\dots,v_n\}$ and arc set $\{v_iv_{i+1} \mid 1 \leqslant i < n\}$.
- S_n – the out-going star of order n, with vertex set $\{v_1,\dots,v_n\}$ and arc set $\{v_1v_i \mid 1 < i \leqslant n\}$.
- RT_n – the rotational tournament of order $2n+1$ with vertex set $\{v_0, v_1, \dots, v_{2n}\}$ and arc set $\{v_iv_{i+j} \mid 0 \leqslant i \leqslant 2n,\ 1 \leqslant j \leqslant n\}$; subscript addition taken modulo $2n+1$.
- T_n – an arbitrary tournament on n vertices.

In order to ensure $r < \infty$ in the definition of the Ramsey number, it is necessary to assume that each D_i is acyclic, for otherwise the transitive tournaments colored completely with color i provide arbitrarily large admissible constructions. On the other hand, if each D_i is acyclic, the basic theorem of Ramsey theory together with the simple fact $r(\mathsf{TT}_n) < 2^{n-1}$ (for all $n > 0$) yields $r < \infty$.

We should note that in the class of tournaments the Ramsey numbers of digraphs even for *one color* are far from being trivial to determine. (This is not the case in many classes of combinatorial structures, e.g. a complete graph obviously contains every smaller complete graph.) The value of $r_1(D)$ has been investigated thoroughly for oriented *trees* D, with special emphasis on the question whether $r_1(D) = |V(D)| - 1$ is valid. (If this equality holds, D is said to be *unavoidable*.) Though it is not even known exactly which orientations of a *path* of length $n-1$ must occur in every T_n, there are many interesting results in this area; see [1, 9, 11, 13, 15, 19–21, 23–25, 29, 30].

On the other hand, for the transitive tournaments there is quite a large gap between the currently best lower and upper bounds

$$c_1 2^{n/2} \leqslant r(\mathsf{TT}_n) \leqslant c_2 2^n,$$

where c_1 and c_2 are positive constants (cf. [6]). Our first observation is that for sufficiently large N, a tournament T_N not only contains TT_n as a subtournament, but also admits a decomposition into transitive parts of order n whenever the trivial divisibility condition $n \mid N$ is satisfied.

Theorem 1. *If $N \geqslant 3 \cdot 4^{n-1}$ and N is a multiple of n, then every tournament of order N can be decomposed into vertex-disjoint copies of TT_n.*

We should emphasize that, contrary to most types of combinatorial structures investigated in [4, 16], no auxiliary ('almost transitive') parts are needed to obtain a complete vertex-decomposition. Alternatively, in the terminology of [8], the class of tournaments has 'Ramsey-remainder' zero for every n.

Ramsey numbers for arc-colored tournaments were first studied for directed *paths* by Gyárfás and Lehel [12] and independently by Bermond [2] and Chvátal [5]. They observed that for directed paths the Ramsey function is multiplicative, i.e.,

$$r(\mathsf{P}_{n_1}, \dots, \mathsf{P}_{n_k}) = \prod_{i=1}^{k} (n_i - 1).$$

In particular, in the diagonal case (with all paths having the same length) we obtain $r_k(\mathsf{P}_n) = (n-1)^k$. (Notice that the case $k=1$ is the simple corollary of Rédei's well known theorem [18] which implies that every tournament contains a directed Hamiltonian path.) Applying a combination of hypergraph-theoretical and probabilistic methods, the second author proved in [27] that a monochromatic P_n appears in an arc-colored tournament of order $(n-1)^k + 1$ even if we only assume that every *vertex* is incident to arcs of at most k distinct colors; i.e., the global assumption on the total number, k, of colors can be replaced by a much weaker local condition.

Investigating a 'more unbalanced' situation, Bialostocki and Dierker [3] observed that the above nice multiplicative property remains valid for the 2-colored 'path vs. transitive tournament' Ramsey numbers as well, i.e.,

$$r(\mathsf{P}_m, \mathsf{TT}_n) = r(\mathsf{P}_m) \cdot r(\mathsf{TT}_n) = (m-1) \cdot r(\mathsf{TT}_n)$$

for all natural numbers m and n. Strengthening the arguments of [3], we shall prove that this formula is a particular case of a more general phenomenon; namely, a similar equality holds whenever a directed path occurs among the digraphs D_i.

Theorem 2. *If $D_k = \mathsf{P}_m$, then*

$$r(D_1, D_2, \ldots, D_k) = (m-1) \cdot r(D_1, \ldots, D_{k-1})$$

holds for arbitrary connected acyclic digraphs $D_1, \ldots, D_{k-1}$.

The condition on connectivity cannot be omitted in general. The simplest counterexample is to take $D_1 = 2\mathsf{TT}_2$, the digraph with four vertices and two vertex-disjoint arcs. Then $r(D_1) = 3$, and it is easy to see that $r(D_1, D_2) \leqslant r(D_2) + 2$ holds for all D_2, including the paths as well.

Not surprisingly at all, it is not true either that the Ramsey numbers for tournaments are (sub- or super-) multiplicative for all connected digraphs. For example, $r(\mathsf{TT}_3) = 3$ and $r_2(\mathsf{TT}_3) = 13 > 3^2$ (cf. [17]), while $r(\mathsf{S}_3) = 3$ and $r_2(\mathsf{S}_3) = 5 < 3^2$.

The latter observation can be generalized: for any collection of stars it is easy to calculate the value of the Ramsey number, as follows.

Proposition 3. *If D_i is a star S_{n_i} for all $1 \leqslant i \leqslant k$, then*

$$r(D_1, \ldots, D_k) = 2(n_1 + \cdots + n_k) - 4k + 1.$$

For the star vs. transitive tournament Ramsey numbers, Bialostocki and Dierker [3] proved the following estimates:

$$r(\mathsf{S}_n) \cdot r(\mathsf{TT}_m) \leqslant r(\mathsf{S}_n, \mathsf{TT}_m) \leqslant (2^{m-1} - 1) \cdot r(\mathsf{S}_n). \tag{1}$$

In particular, for $m=3$ and $m=4$ the two bounds coincide, that is, $r(\mathsf{S}_n, \mathsf{TT}_3) = 6n - 9$ and $r(\mathsf{S}_n, \mathsf{TT}_4) = 14n - 21$. We shall prove that the *lower bound is tight* in (1) for every m and n. This will follow from the next two results.

Theorem 4. *If $D_1,\dots,D_t$ are stars, then*

$$r(D_1,\dots,D_k)\leqslant r(D_1,\dots,D_t)\cdot r(D_{t+1},\dots,D_k)$$

for arbitrary acyclic digraphs $D_{t+1},\dots,D_k$. Moreover, if equality holds, then the extremal tournaments for $D_1,\dots,D_k$ (i.e., k-colored, with $r(D_1,\dots,D_k)$ vertices, where no D_i appears in color i, for any $1\leqslant i\leqslant k$) can be obtained from those for $D_1,\dots,D_t$ and for $D_{t+1},\dots,D_k$, by the operation of substitution.

A more precise description of the colorings attaining equality will be given in the proof of the theorem, in Section 3.

Proposition 5. *If D_k is a transitive tournament, then*

$$r(D_1,\dots,D_k)\geqslant r(D_1,\dots,D_{k-1})\cdot r(D_k)$$

for arbitrary connected acyclic digraphs $D_1,\dots,D_{k-1}$.

Again, connectivity may be necessary. To see this, we take the example of $D_1=2\mathsf{TT}_2$ and $D_2=\mathsf{S}_3$. Then $r(D_1,D_3)\leqslant r(D_3)+2=5<9=r(D_1)\cdot r(D_2)$.

Combining the above two propositions, we immediately obtain

Corollary 6. *For every $n\geqslant 3$ and $m\geqslant 3$, $r(\mathsf{S}_n,\mathsf{TT}_m)=r(\mathsf{S}_n)\cdot r(\mathsf{TT}_m)$.*

Letting n tend to infinity, an asymptotic version of Corollary 6 becomes valid in a very general form. To state it explicitly, we need to introduce some notation.

Definition. Let D and H be digraphs. We say that H contains a homomorphic image of D, denoted $D<H$, if there is a mapping $\eta: V(D)\to V(H)$ such that $uv\in A(D)$ implies $\eta(u)\eta(v)\in A(H)$. We denote by $q(D)$ the largest integer q for which there exists a tournament of order q containing no homomorphic image of D. More generally, if $D_1,\dots,D_k$ are acyclic digraphs, $q(D_1,\dots,D_k)$ denotes the largest integer q for which there exists a tournament $\mathsf{T}=(V,A)$ of order q with an arc-coloring $\varphi: A\to\{1,\dots,k\}$ such that no color class i contains a homomorphic image of D_i.

For an arbitrary digraph D, the inequality $q(D)\leqslant r(D)$ holds by definition, and also

$$q(D)\leqslant r(\mathsf{TT}_{p(D)})$$

is valid, where $p(D)$ denotes the maximum number of vertices in a directed path of D. In particular, $q(D)=1$ if and only if $p(D)=2$. The importance of the function q in connection with Ramsey numbers is shown by the following result.

Theorem 7. *For arbitrary acyclic digraphs* $D_1,\ldots,D_k$,

$$r(D_1,\ldots,D_k,\mathsf{S}_n) = 2n \cdot q(D_1,\ldots,D_k) + o(n)$$

as $n \to \infty$.

One interesting aspect of this theorem is that if the D_i are supposed to contain no pair of consecutive arcs (that is, P_3 is excluded), then the actual choice of those digraphs has no effect on the asymptotic behavior of the Ramsey number as it is equal to $2n + o(n)$. (The D_i can only influence the 'error term' $o(n)$.)

The proofs are presented in Sections 2–4. In the concluding section we discuss some problems that remain open.

2. Recursive constructions and the path reduction theorem

The main concern of this section is to show how the Ramsey numbers can be computed recursively if directed paths are involved among the D_i (see Theorem 2). As regards 'multiplicative' *lower* bounds, the following operation called *substitution* will be useful. Let T' and T'' be arc-colored tournaments, and assume that no color appears in both of them. The tournament $\mathsf{T}''[\mathsf{T}']$ is obtained by replacing each vertex v of T'' by a tournament $\mathsf{T}(v)$ having the same orientation and coloring as T', and the orientation and color of the arcs joining $\mathsf{T}(v)$ and $\mathsf{T}(v')$ in $\mathsf{T}''[\mathsf{T}']$ are the same as those of the arc joining v and v' in T''.

Our first example where the operation of substitution turns out to be useful is given in the following assertion.

Lemma 8. *Let* $D_1,\ldots,D_t$ *be connected acyclic digraphs, and* $D_{t+1},\ldots,D_k$ *acyclic. Then*

$$r(D_1,\ldots,D_k) \geqslant r(D_1,\ldots,D_t) \cdot q(D_{t+1},\ldots,D_k).$$

Proof. We take tournaments $\mathsf{T}' = (V',A')$ and $\mathsf{T}'' = (V'',A'')$ with arc-colorings $\varphi' : A' \to \{1,\ldots,t\}$ and $\varphi'' : A'' \to \{t+1,\ldots,k\}$, $|V'| = r(D_1,\ldots,D_t)$ and $|V''| = q(D_{t+1},\ldots,D_k)$, such that no D_i occurs in color i for $1 \leqslant i \leqslant t$ in T', and T'' contains no homomorphic image of any D_j in color j for $t+1 \leqslant j \leqslant k$. Then $\mathsf{T}''[\mathsf{T}']$ has order $r(D_1,\ldots,D_t) \cdot q(D_{t+1},\ldots,D_k)$ and contains no D_i in color i, for all $1 \leqslant i \leqslant k$, since the copies of T' are vertex-disjoint (yielding the required property for $1 \leqslant i \leqslant t$) while any $D_j \subset \mathsf{T}''[\mathsf{T}']$ in color j $(t+1 \leqslant j \leqslant k)$ would imply $D_j < \mathsf{T}''$ by contracting the vertex classes of the substitution. $\square$

Though the function q is just a lower bound on r in general, in some cases the two values coincide. The next lemma exhibits two such examples.

Lemma 9. *For every* $n \geqslant 3$, $q(\mathsf{P}_n) = r(\mathsf{P}_n) = n-1$ *and* $q(\mathsf{TT}_n) = r(\mathsf{TT}_n)$.

Proof. Since TT_{n-1} contains no homomorphic image of P_n, we clearly have $n - 1 \leqslant q(\mathsf{P}_n) \leqslant r(\mathsf{P}_n) = n-1$, i.e., equality must hold throughout. On the other hand, every homomorphic image of TT_n is in fact isomorphic to TT_n because no two of its vertices can be mapped onto the same vertex. □

Proof of Proposition 5. Combining Lemmas 8 and 9, the inequality $r(D_1,\dots,D_k) \geqslant r(D_1,\dots,D_{k-1}) \cdot r(D_k)$ follows immediately whenever D_k is transitive. □

We now prove that the multiplicative rule applies when $D_k = \mathsf{P}_m$ is a directed path.

Proof of Theorem 2. Denote $r = r(D_1,\dots,D_{k-1})$. The lower bound $r(D_1,\dots,D_k) \geqslant r \cdot (m-1)$ again, as in the previous proof, follows by Lemmas 8 and 9. (In the present case, we take a tournament T' on $r(D_1,\dots,D_{k-1})$ vertices, with a $(k-1)$-coloring on its arcs, containing no D_i in any color i, and substitute it into the transitive tournament $\mathsf{T}'' = \mathsf{TT}_{m-1}$ of color k. Then $\mathsf{T}''[\mathsf{T}']$ has order $r \cdot (m-1)$ and contains no D_i for $1 \leqslant i \leqslant k$.)

To prove the upper bound $r(D_1,\dots,D_k) \leqslant r \cdot (m-1)$, suppose that $\mathsf{T} = (V, A)$ is a k-colored tournament on more than $r \cdot (m-1)$ vertices. Define $G = (V, E)$ as the graph on the same vertex set, whose edges are the pairs of vertices adjacent by an arc of color k in T.

If G has an independent set of more than r vertices, then T contains a $(k-1)$-colored subtournament of order $r+1$, hence a D_i occurs in color i for some $i \leqslant k-1$. Otherwise, if the independence number of G is at most r, we obtain that the chromatic number of G is at least m. Thus, by the Gallai–Roy theorem [10, 22] (see also [28]), every orientation of G contains a directed path of length $m-1$, and therefore P_m is a subgraph of T in color k. □

3. Stars

In this section we prove the results involving stars, namely Proposition 3, Theorems 4 and 7.

Proof of Proposition 3. A simple construction showing the lower bound

$$r(\mathsf{S}_{n_1},\dots,\mathsf{S}_{n_k}) \geqslant 2n_1 + \dots + 2n_k - 4k + 1$$

is to consider the rotational tournament RT_N of order $2N+1$ with $N = n_1 + \cdots + n_k - 2k$, take an arbitrary partition of the N arcs starting at some (fixed) vertex into k classes such that the ith class consists of $n_i - 2$ arcs, and rotate this 'partial coloring' to obtain a partition of the arcs starting at each vertex of RT_N. Alternatively, one can begin with a decomposition of the edge set of the complete graph on $2(n_1 + \cdots + n_k - 2k) + 1$

vertices into $n_1 + \cdots + n_k - 2k$ Hamiltonian cycles, assign a color to each cycle (color i should appear on $n_i - 2$ cycles) and take a cyclic orientation in every cycle.

Conversely, if T has more than $2(n_1 + \cdots + n_k - 2k) + 1$ vertices, then some vertex v has out-degree greater than $n_1 + \cdots + n_k - 2k$. Thus, in every k-coloring of T there is some i such that there are more than $n_i - 2$ arcs of color i starting at v, by the pigeon-hole principle. Hence, v is the center of S_{n_i} in color i. $\square$

Proof of Theorem 4. From the argument proving the inequality $r(D_1, \ldots, D_k) \leqslant r(D_1, \ldots, D_t) \cdot r(D_{t+1}, \ldots, D_k)$ it will also turn out that if equality holds, then the extremal k-colorings on $r(D_1, \ldots, D_k)$ vertices are obtained as a more general variant of substitutions. Namely, the first t color classes have to form $r(D_{t+1}, \ldots, D_k)$ vertex-disjoint tournaments of order $r(D_1, \ldots, D_t)$ each, i.e., that many (possibly different) extremal t-colorings are substituted into an extremal $(k - t)$-coloring.

Suppose that $\mathsf{T} = (V, A)$ is a k-colored tournament on $n > r \cdot r(D_{t+1}, \ldots, D_k)$ vertices, where $r = r(D_1, \ldots, D_t)$. If T contains a star D_i in some color i $(1 \leqslant i \leqslant t)$, then we have nothing to prove. Otherwise, consider the graph $G = (V, E)$ (on the vertex set of T) where two vertices are adjacent if and only if they are joined by an arc of T in some color i, $1 \leqslant i \leqslant t$. In the orientation induced by T in G, every vertex has out-degree at most $\frac{1}{2}(r - 1)$ (cf. Proposition 3), therefore the *average degree*, say d, of G is at most $r - 1$. Applying Turán's theorem [26], the vertex-independence number $\alpha(G)$ of G has the following lower bound:

$$\alpha(G) \geqslant \frac{n}{d+1} \geqslant \frac{n}{r} > r(D_{t+1}, \ldots, D_k). \tag{2}$$

Such a large independent set of G induces a $(k - t)$-colored subtournament in T, therefore a subdigraph D_i must occur in some color i, $t + 1 \leqslant i \leqslant k$.

If there are only $n = r \cdot r(D_{t+1}, \ldots, D_k)$ vertices in T, all the four expressions in (2) may be equal. In the case of equality, however, G has to be the vertex-disjoint union of *complete* graphs of order r (again by Turán's theorem), and then each component of G has to induce a t-colored subtournament which is extremal for $D_1, \ldots, D_t$. $\square$

Proof of Theorem 7. If the graphs $D_1, \ldots, D_k$ are fixed, then $q(D_1, \ldots, D_k)$ is a constant. Therefore, and since S_n is connected, the inequality $r(D_1, \ldots, D_k, \mathsf{S}_n) \geqslant r(\mathsf{S}_n) \cdot q(D_1, \ldots, D_k) = (2n - 3) \cdot q(D_1, \ldots, D_k) = 2n \cdot q(D_1, \ldots, D_k) - \mathrm{O}(1)$ follows by Lemma 8 as $n \to \infty$.

To prove an upper bound of the same asymptotic behavior, we apply an argument that also uses some ideas from the proof of Theorem 4. We write q to abbreviate $q(D_1, \ldots, D_k)$. For a contradiction, suppose that there is a positive constant c with the following property: for arbitrarily large values of n there exist tournaments T_N on $N = 2n \cdot (q + c)$ vertices, with a $(k + 1)$-coloring such that no D_i occurs in color i $(1 \leqslant i \leqslant k)$ and color $k + 1$ contains no S_n. The latter assumption means that the out-degrees in color $k + 1$ are smaller than n (and also smaller than $n - 1$, but we do not need this stronger fact). Hence, the $(k + 1)$th color class consists of at most $(n - 1)N$

edges. Therefore, the total size of the first k color classes is at least

$$\binom{N}{2} - (n-1)N > \frac{1}{2}N(N-2n) = \frac{1}{2}\frac{q-1+c}{q+c}N^2 \geqslant \frac{q-1}{2q}(1+\varepsilon)N^2$$

for some $\varepsilon = \varepsilon(c,q) > 0$. We denote by G the graph formed by those edges.

Applying the Erdős–Stone theorem [7], we obtain that for every t there is a sufficiently large N ensuring the existence of a complete $(q+1)$-partite subgraph $H \subset G$, with t vertices in each class of H. We define t to be 'very large', which means

$$t = a^{a^{\cdot^{\cdot^{\cdot^{a^m}}}}},$$

where $a = a(k)$ is an appropriate constant to be chosen later, $m = \max\{|V(D_i)| \mid 1 \leqslant i \leqslant k\}$, and the height of the tower is $q+1$ or $q+2$ (whichever is even).

Denote by $A_1, \ldots, A_{q+1}$ the vertex classes of H, and consider a complete graph K_{q+1} with vertex set $\{1, \ldots, q+1\}$. As is well known, the edge set of K_{q+1} can be partitioned into q perfect matchings if q is odd, or into $q+1$ matchings of $q/2$ edges each if q is even. We fix one such edge partition of K_{q+1} and denote its edge classes by $F_1, F_2, \ldots$. Those matchings F_j can be associated with matchings on the classes of H in such a way that the edge (ℓ, ℓ') of F_j corresponds to the pair (A_ℓ, A'_ℓ). (In particular, if $q(D_1, \ldots, D_k) = 1$, then the unique edge of K_2, viewed as matching F_1, represents the pair (A_1, A_2) in H).

Taking the matchings F_1, $F_2, \ldots$ one by one, we proceed as follows. According to color and orientation, the arcs joining A_ℓ and $A_{\ell'}$ form $2k$ classes in each pair of matched parts in H. The class with the largest number of arcs has edge density at least $1/2k$, therefore it has to contain a fairly large complete bipartite subgraph. More explicitly, the possible size of this subgraph can be estimated from below by the results of Kővári et al. [14]: as k is fixed, we can select subsets $A \subset A_\ell$ and $A' \subset A_{\ell'}$ with $|A| = |A'| \geqslant c' \log t$ for some constant $c' = c'(k) > 0$, where all arcs have the same color and are oriented in the same direction between A and A'. For each $j = 1, 2, \ldots, q$ or $q+1$, we reduce all classes of H to subsets of the same cardinality, say t', such that each matching pair under F_j satisfies this property of homogenity. (If q is even, then the unmatched class is reduced to an arbitrary subset of cardinality t').

Notice that for an appropriately chosen value of a, we can ensure

$$t' = a^{a^{\cdot^{\cdot^{\cdot^{a^m}}}}},$$

where the height of the tower for t' is just one less than that for t. Applying this operation q or $q+1$ times, we eventually obtain $q+1$ vertex classes $B_1, \ldots, B_{q+1}$ of cardinality m each ($B_j \subset A_j$ for all $1 \leqslant j \leqslant q+1$), where each pair of classes induces a monochromatic complete bipartite directed graph all of whose arcs are oriented in the same direction. By the definition of q, contracting the classes B_j to single vertices x_j, the k-colored tournament obtained on $q+1$ vertices has to contain a homomorphic image D' of D_i in color i, for some $1 \leqslant i \leqslant k$. Since D_i has at most m vertices, the existence of D' immediately implies that D_i is a subdigraph of T_N in color i. This contradiction completes the proof. $\square$

4. Decompositions

In this short section we prove Theorem 1. The assertion is obvious for $n \leqslant 2$, hence we consider $n \geqslant 3$ only.

Let $\mathsf{T} = \mathsf{T}_N$ be an arbitrary tournament on $N \geqslant 3 \cdot 4^{n-1}$ vertices, and assume that N is a multiple of n. Denoting $r = r(\mathsf{TT}_n)$, recall that $r < 2^{n-1}$ holds, therefore $r(\mathsf{TT}_{2n-1}) < 4^{n-1}$. Hence, we can successively select r subsets $V_1, V_2, \ldots, V_r \subset V$ of cardinality $2n-1$ each, such that every V_i $(1 \leqslant i \leqslant r)$ induces a transitive subtournament of T. (Having selected the first at most $r-1$ sets V_i, we have taken fewer than $n \cdot 2^n < 2 \cdot 4^{n-1}$ vertices as $n \geqslant 3$; thus, more than $4^{n-1} > r(\mathsf{TT}_{2n-1})$ vertices remain to choose from.)

In the next step, we choose $N/n - 2r$ mutually vertex-disjoint copies $\mathsf{T}^1, \mathsf{T}^2, \ldots, \mathsf{T}^{N/n-2r}$ of TT_n in $V' := V \backslash (V_1 \cup \cdots \cup V_r)$. This can be done, again successively, since as long as fewer than $N/n - 2r$ subtournaments are selected, we have at least $N - r \cdot (2n-1) - n \cdot (N/n - 2r - 1) = r + n > r$ vertices to consider. At the end of this process, precisely r vertices of V' remain uncovered; call them $v_1, \ldots, v_r$. Each v_i either dominates or is dominated by n vertices of V_i, therefore each $V_i \cup \{v_i\}$ can be partitioned into two transitive tournaments of order n. Thus, those pairs together with the T^j $(1 \leqslant j \leqslant N/n - 2r)$ form a decomposition of T with the required properties.

5. Concluding remarks and open problems

1. *Multiplicativity and substitutions.* We have shown in the proof of Theorem 4 that the Ramsey number of a set of stars vs. a set of other digraphs is not only multiplicative but also every extremal configuration is obtained by (a general type of) substitution. It would be interesting to find further combinations of digraphs where the Ramsey function satisfies this strong property which we shall call 'structural multiplicativity'.

A closely related basic open problem remains to characterize the class of combinations of digraphs for which the Ramsey function is multiplicative.

It is important to note that multiplicativity alone does not imply structural multiplicativity. Consider, for instance, the paths, where we know that $r(\mathsf{P}_{n_1}, \ldots, \mathsf{P}_{n_k}) = \prod_{i=1}^{k} r(\mathsf{P}_{n_i}) = \prod_{i=1}^{k}(n_i - 1)$ holds. It can be shown (by an argument related to the proof of Theorem 2) that an edge coloring of the complete graph $K = K_{(n_1-1)\cdots(n_k-1)}$ admits a tournament orientation with no P_{n_i} in any color i if and only if each edge class E_i (i.e., of color i) forms a graph G_i of chromatic number $n_i - 1$. This condition also implies that the corresponding vertex partitions (proper colorings) in the G_i have a nice structure: the vertices of K can be represented by the integral lattice points of the k-dimensional box B with respective side lengths $n_i - 2$, and the $n_i - 1$ layers of B

orthogonal to the ith coordinate axis are the vertex classes in a proper (n_i-1)-coloring of G_i. This orthogonal structure of the k vertex-partitions, however, still admits much freedom to decide about the colors of the edges, as only the edges parallel to any one coordinate axis must have the same color. Hence, some of the tournaments extremal for $\mathsf{P}_{n_1},\ldots,\mathsf{P}_{n_k}$ are obtained by substitution, while some others (most of them, actually) are not.

2. *Decompositions*. Probably, the bound $N \geqslant 3\cdot 4^{n-1}$ in Theorem 1 can be weakened to a much smaller value. Independently of the (actually unknown) growth rate of $r(\mathsf{TT}_n)$, we expect that the condition $N \geqslant (1-\mathrm{o}(1))\cdot(r(\mathsf{TT}_n))^{2-\varepsilon}$ is sufficient for some $\varepsilon>0$ as $n\to\infty$.

3. *Monochromatic transitive tournaments*. It would be of great interest to determine the value of $\lim_{n\to\infty}(1/n)\log r_1(\mathsf{TT}_n)$, or even to prove that the limit exists. Such a result, however, seems to be beyond the power of the methods currently available, similarly to the analogous problems on the Ramsey numbers of (undirected) complete graphs.

4. *Digraph vs. undirected graph Ramsey numbers*. It would be worth finding strong links between pairs of directed and undirected versions of (k-color) Ramsey numbers. For example, investigate how much $r_k(\mathsf{TT}_3)$ can exceed the (diagonal) Ramsey number for triangles, that is the largest integer $r=r(3,\ldots,3)$ such that the complete graph K_r admits an edge coloring with k colors in which no monochromatic K_3 occurs. Also, independently of the value of $r(3,\ldots,3)$ (which is quite hard to estimate), one should find ‘reasonable’ bounds on $r_k(\mathsf{TT}_3)$.

A related question, relevant in connection with Theorem 7, is the case $q(D_1,\ldots,D_k)=1$. This means that each D_i is bipartite, and all arcs are oriented from one vertex class to the other. Perhaps these restricted structures admit tight bounds on $r(D_1,\ldots,D_k)$ in terms of the undirected Ramsey numbers $r(G_1,\ldots,G_k)$ where G_i is the underlying graph of D_i. An interesting particular question is what happens with *paths D_i whose orientations contain no* P_3.

5. *Trees*. Concerning the monochromatic Ramsey numbers of oriented trees, one of the most challenging open problems is Sumner’s conjecture (see [19]) stating that $r_1(T)\leqslant 2n-3$ always holds if the tree T has n vertices. Also, as we have already mentioned, it seems to be quite difficult to characterize the ‘unavoidable’ trees of order n, i.e., those for which $r_1(T)=n-1$, even in the particular case where the underlying graph of T is a path.

One might also investigate the growth of the diagonal Ramsey numbers $r_k(T)$ as a function of the number k of colors, where the tree T is fixed and k gets large.

6. *Related structures*. As far as we know, no similar problems have been studied for related classes of digraphs other than the complete symmetric ones. Natural candidates for future research are for example those subdigraphs of bipartite tournaments which contain no directed paths of length two. In this case one can expect strong relations with the corresponding undirected Ramsey numbers defined for bipartite graphs (in edge colorings of $K_{n,n}$). Such investigations would be analogues of the ones proposed in the paragraphs of 4. above.

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